Traffic Generation Concepts

Random Arrivals

By

James F. Brady

Inquiries

(mailto:JmsFBrdy@aol.com)

CONTENTS

TABLES AND FIGURES

1. INTRODUCTION

This document is intended to summarize the mechanics of offering independent request traffic to a system when the number of traffic sources is very large and can be assumed to be infinite. Systems with this independent request property can be found in many places including telecommunications networks and computer systems. In fact, the telephone network was originally designed and implemented using this random arrivals assumption.

What follows is a brief description of the random arrivals traffic environment and a presentation of the formulas used to artificially produce traffic consistent with that environment. Motivation for using the presented formulas is provided through a mathematical derivation and a supporting example.

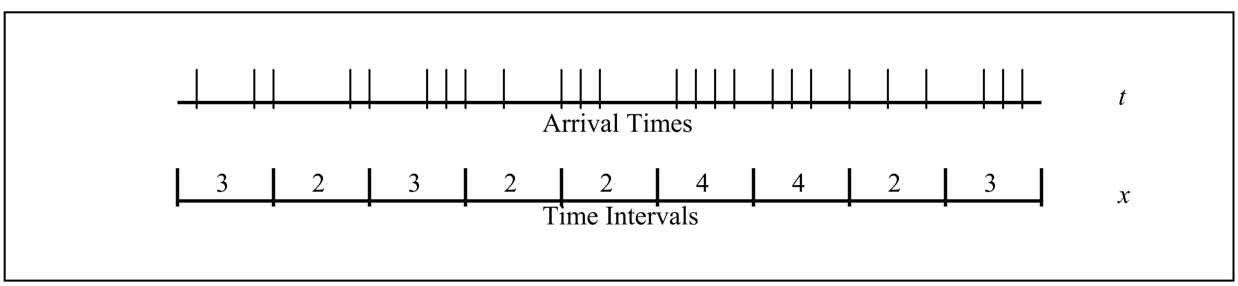
2. RANDOM ARRIVALS

From the being of the telecommunications industry, it has been understood that as a population telephone users act independently when deciding when to make a call. It is further understood that these users have biases regarding the time of day or day of week chosen to make a call. These two understandings give rise to the notion that network accesses, arrivals, are assumed to be independent from each other and to occur at a constant rate during the time period of interest, e.g., the busy hour. This arrival pattern is referred to in the literature as a Poisson process and is quantified by two formulas from probability theory. The times between these random arrivals (t) are negative-exponentially distributed and the number of arrivals in constant length intervals is Poisson distributed.

Figure 1, below, is a pictorial representation of a random arrivals pattern where the arrival times, *t*, are represented by the non-uniform vertical bars and the frequency counts, *x*, are the numbers indicating the arrivals within the uniform intervals shown. Mathematically (t), is represented by the negative-exponential probability density function, Equation 2.1, which has a mean inter-arrival time of μ . The number of arrivals (x) in constant length intervals is mathematically illustrated as the Poisson probability density function, shown as Equation 2.2, which has a mean number of arrivals per interval of λ .

It is interesting to note that the mean of the negative-exponential distribution is equal to its standard deviation and the mean of the Poisson distribution is equal to its variance. Formulas for estimating mean, standard deviation, and variance are provided at the end of this section. These parameter estimation formulas can be used to analyze arrival data and make judgments concerning the likelihood the arrivals are random.

Figure 1: Random Arrivals



$$f(t) = \frac{1}{\mu} e^{-\frac{t}{\mu}} \tag{2.1}$$

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \tag{2.2}$$

Where:

$\mu = mean\ int\text{–}arrival\ time\ t$ $\quad$ $For\ f(t),\ \ \mu = mean = variance$

$\lambda = mean\ number\ of\ arrivals\ per\ interval\ x$ $\quad$ $For\ f(x),\ \ \lambda = mean = std\ dev = \sqrt{variance}$

Estimates of mean, variance, and standard deviation are:

$$mean\ estimate\ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n},\ variance\ estimate\ s^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n-1},\ std\ dev\ estimate\ s = \sqrt{variance}$$

3. TRAFFIC GENERATION FORMULA

In the previous section, the traffic pattern simulating random arrivals is described and in this section the formulas needed to produce this traffic pattern are defined. Either Equation 3.1 or Equation 3.2, below, is used to construct times between arrivals that adhere to the random arrivals requirement above. Applying Equation 3.2, the inter-arrival time, t_0, is obtained by multiplying minus the mean inter-arrival time, $-\mu$, by the natural logarithm of a random number between zero and one, $ln(r_0)$. Equation 3.2 is generally used over Equation 3.1 to obtain values of t_0 because $t_0 = -\mu\, ln(1 - r_0)$ and $t_0 = -\mu\, ln(r_0)$ yield symmetrical results and Equation 3.2 requires one less arithmetic operation than does Equation 3.1.

$$t_0 = -\mu\, ln(1 - r_0) \tag{3.1}$$

$$t_0 = -\mu\, ln(r_0) \tag{3.2}$$

Where:

$t_0 = time\ until\ the\ next\ arrival$ $\quad$ $ln = natural\ log\ (ln\, e = 1)$

$\mu = mean\ inter\text{–}arrival\ time$ $\quad$ $r_0 = random\ number\ 0 \le r_0 \le 1$

Motivation for use of the above equations for obtaining inter-arrival times is contained in the example application and mathematical derivation that follow.

4. TRAFFIC GENERATION GRAPHICAL ILLUSTRATION

This and the next section provide motivation for the use of Equation 3.1 or Equation 3.2 for obtaining independent inter-arrival times by using a combination of mathematical rigor and an example with the mean inter-arrival time, $\mu = 5$. The mathematical rigor is contained in the next section, Section 5, and the corresponding example is included in the spreadsheet graphs and tables of Figure 2 below.

The technique used to produce the Negative-Exponentially distributed inter-arrival times is the Cumulative Distribution Function reverse transformation technique implemented in many simulation software packages.

The first step in this approach is to integrate the Negative-Exponential Probability Density Function over the range $(0, t_0)$ as shown in Equation 5.1 through Equation 5.9. The second step is to equate the results of this integral to a (0,1) random number, r_0, as detailed in Equations 5.10 through Equation 5.16, and solve for t_0, the desired time to the next arrival.

The $F(t)$ portion of the Figure 2 graph below illustrates this reverse transformation technique graphically. A random number (0,1) is drawn and its location is found on the F(t) axis. Then, a horizontal line is drawn from the F(t) axis until it intersects the desired curve, F(t) for $(1-r_0)$ or 1 – F(t) for r_0. The delay time until the next arrival is the t axis value at that intersection point.

Figure 2: TRAFFIC GENERATION GRAPHICAL ILLUSTRATION

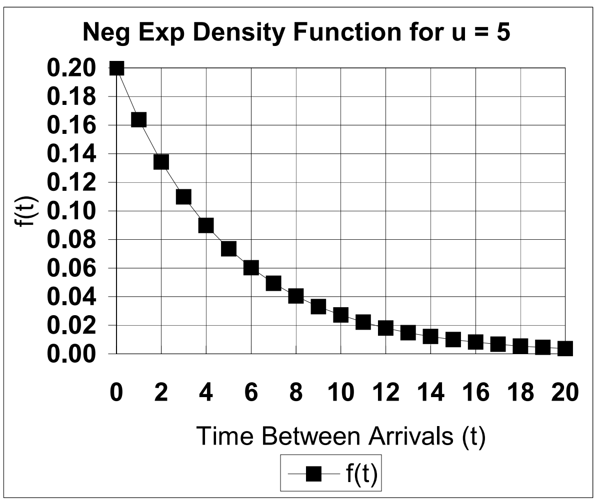
$f(t) = Neg\ Exp\ Dist\ Density\ Function$

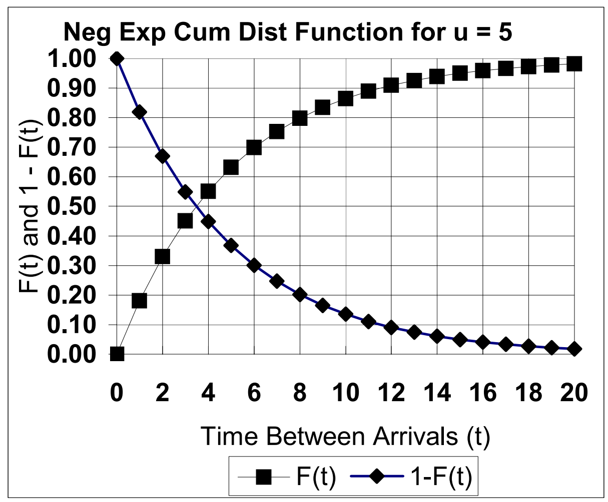
$F(t) = Neg\ Exp\ Cum\ Dist\ Function$

$$f(t) = \frac{1}{\mu} e^{-\frac{t}{\mu}}$$

$$F(t) = \int_0^t f(t)\, dt$$

Random Arrivals - Generation Of Negative-Exponentially Distributed Random Numbers





u = 5

t	f(t)	F(t)	1-F(t)	t=-u*ln(1-colC)	t=-u*ln(colD)
0	0.2000	0.0000	1.0000	0	0
1	0.1637	0.1813	0.8187	1	1
2	0.1341	0.3297	0.6703	2	2
3	0.1098	0.4512	0.5488	3	3
4	0.0899	0.5507	0.4493	4	4
5	0.0736	0.6321	0.3679	5	5
6	0.0602	0.6988	0.3012	6	6
7	0.0493	0.7534	0.2466	7	7
8	0.0404	0.7981	0.2019	8	8
9	0.0331	0.8347	0.1653	9	9
10	0.0271	0.8647	0.1353	10	10
11	0.0222	0.8892	0.1108	11	11
12	0.0181	0.9093	0.0907	12	12
13	0.0149	0.9257	0.0743	13	13
14	0.0122	0.9392	0.0608	14	14
15	0.0100	0.9502	0.0498	15	15
16	0.0082	0.9592	0.0408	16	16
17	0.0067	0.9666	0.0334	17	17
18	0.0055	0.9727	0.0273	18	18
19	0.0045	0.9776	0.0224	19	19
20	0.0037	0.9817	0.0183	20	20

5. TRAFFIC GENERATION FORMULA MATHEMATICAL DERIVATION

This section is intended to provide a detailed derivation of Equation 3.1 and Equation 3.2, which are used to generate delay time values that represent random arrivals. Observe that Equation 5.15 and Equation 5.16, below, are the same as Equation 3.1 and Equation 3.2.

Integrate the Probability Density Function over the range $(0, t_0)$:

$$f(t) = \frac{1}{\mu} e^{-\frac{t}{\mu}} \tag{5.1}$$

$$F(t_0) = \int_0^{t_0} f(t)\, dt \tag{5.2}$$

$$F(t_0) = \int_0^{t_0} \frac{1}{\mu} e^{-\frac{t}{\mu}}\, dt \tag{5.3}$$

$$F(t_0) = \frac{1}{\mu} \int_0^{t_0} e^{-\frac{t}{\mu}} dt \tag{5.4}$$

From Calculus:

$$\int e^{at} dt = \frac{e^{at}}{a} \tag{5.5}$$

Evaluate the Cum Dist Function at its limits $(0, t_0)$:

$$F(t_0) = \frac{1}{\mu} \left[-\mu e^{-\frac{t}{u}} \right]_0^{t_0} \tag{5.6}$$

$$F(t_0) = \frac{1}{\mu} \left[\left(-\mu e^{-\frac{t_0}{\mu}} \right) - \left(-\mu e^{0} \right) \right] \tag{5.7}$$

$$F(t_0) = \frac{1}{\mu} \left[\mu - \mu\, e^{-\frac{t_0}{\mu}} \right] \tag{5.8}$$

$$F(t_0) = 1 - e^{-\frac{t_0}{\mu}} \tag{5.9}$$

Set: $r_0 = F(t_0)$ and solve for t_0

$$r_0 = F(t_0) = 1 - e^{-\frac{t_0}{\mu}} \tag{5.10}$$

$$r_0 = 1 - e^{-\frac{t_0}{\mu}} \tag{5.11}$$

$$e^{-\frac{t_0}{\mu}} = 1 - r_0 \quad (5.12)$$

$$ln\left(e^{-\frac{t_0}{\mu}}\right) = ln(1 - r_0) \quad (5.13)$$

$$-\frac{t_0}{\mu} = ln(1 - r_0) \quad (5.14)$$

$$t_0 = -\mu \, ln(1 - r_0) \quad (5.15)$$

Given the symmetry of $(1 - r_0)$ *and* r_0

$$t_0 = -\mu \; ln(r_0) \quad (5.16)$$

6. SUMMARY

This document describes the mechanics of creating random arrivals traffic when the number of traffic sources is infinite. It contains a description of the random arrivals traffic pattern, lists the set of formulas required to produce random arrivals, and provides the mathematical derivation of the random arrivals formulas.

7. AUTHOR

James F. Brady received both BS (1971) and MS (1973) degrees in Industrial and Systems Engineering from The Ohio State University. Mr. Brady has worked in the telecommunications and computer industries for such companies as GTE, Tandem Computers, and Siemens Corporation. In 1984 he authored the Traffic Grade of Service Standards referenced in GTE's Federal Access Tariff as that company's telephone network traffic sizing rules. More recently he is the inventor named on Siemens US/EU Patent (2002E14421US / 03004440.8-). This patent is for a telephony based computer response time congestion control invention that implements Exponential Smoothing forecasting techniques. Mr. Brady is currently the Capacity Planner for the State Of Nevada within the Department of Information Technology.